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Transitivity of Γ-relation on hyperfields

by
S. Mirvakili, S.M. Anvariyeh and B. Davvaz

Abstract

In this paper we introduce the complete parts on hyperrings and study the complete closure on hyperrings. Also, we consider the fundamental relation Γ on hyperrings and we prove that the relation Γ is transitive on hyperfields.

Key Words: Hyperring, complete part, hyperfield, strongly regular relation, fundamental relation, transitive.
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1 Introduction

A *hypergroupoid* $(H, \circ)$ is a non-empty set H together with a *hyperoperation* $\circ$ defined on H, that is, a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y \mid x \in A,\ y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *hypergroup* in the sense of Marty [10] if for all $x, y, z \in H$ the following two conditions hold: (i) $x \circ (y \circ z) = (x \circ y) \circ z$, (ii) $x \circ H = H \circ x = H$, means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$ and $y \in v \circ x$. If $(H, \circ)$ satisfies only the first axiom, then it is called a *semi-hypergroup.* An exhaustive review updated to 1992 of hypergroup theory appears in [1]. A recent book [2] contains a wealth of applications.

If H is a semi-hypergroup and $\rho \subseteq H \times H$ is an equivalence relation then for all pairs (A, B) of non-empty subsets of H, we set $A\overline{\overline{\rho}}B$ if and only if $a\rho b$ for all $a \in A$ and $b \in B$. The relation ρ is said to be *strongly regular to the right* if $x\rho y$ implies $x \circ a\ \overline{\overline{\rho}}\ y \circ a$ for all $(x, y, a) \in H^3$. Analogously, we can define *strongly regular to the left.* Moreover ρ is called *strongly regular* if it is strongly regular to the right and to the left. Let H be a hypergroup and ρ an equivalence relation on H. Let $\rho(a)$ be the equivalence class of a with respect to ρ and let $H/\rho = \{\rho(a) \mid a \in H\}$.

A hyperoperation $\otimes$ is defined on H/ρ by $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in \rho(a) \circ \rho(b)\}$. If ρ is strongly regular then it readily follows that $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in a \circ b\}$. It is well known for ρ strongly regular that $< H/\rho, \otimes >$ is a group (see Theorem 31 in [1]), that is $\rho(a) \otimes \rho(b) = \rho(c)$ for all $c \in a \circ b$. a *hyperring* [15] is a multi-valued system $(R, +, \circ)$ which satisfies the ring-like axioms in the following way: (i) $(R, +)$ is a hypergroup in the sense of Marty, (ii) $(R, \circ)$ is a semi-hypergroup, (iii) The multiplication is distributive with respect to the hyperoperation $+$. The hyperrings were studied by many authors, for example see[4,6,12].

The fundamental relation β^* was introduced on hypergroups by Koskas [9], and studied by many authors, for example see [3,5,11,14]. The fundamental relation β^* is defined on hypergroups as the smallest equivalence relation so that the quotient would be a group. Let H be a hypergroup and U be the set of all finite products of elements of H and define the relation β on H as follows:

$$x\beta y \text{ if and only if } \{x, y\} \subseteq u \text{ for some } u \in U.$$

For hypergroups we have $\beta^* = \beta$. Freni turn his research on the direction to find classes on semi-hypergroups, such that the above equality works. Among others he defined the γ-relation on semi-hypergroups and hypergroups, see [7] and studied by many authors, for example see [5,8]. We recall the following definition from [7]. If H is a hypergroup, then we set: $\gamma_1 = \{(x, x) \mid x \in H\}$ and, for every integer $n > 1$, γ_n is the relation defined as follows:

$$x\gamma_n y \iff \exists (z_1, z_2, \ldots, z_n) \in H^n, \exists \sigma \in \mathbb{S}_n : \; x \in \prod_{i=1}^{n} z_i, \quad y \in \prod_{i=1}^{n} z_{\sigma(i)}.$$

Obviously, for every $n \geq 1$, the relations γ_n are symmetric, and the relation $\gamma = \cup_{n\geq 1}\gamma_n$ is reflexive and symmetric. Let γ^* be the *transitive closure* of γ. In [7], it is proved that γ^* is the smallest strongly regular equivalence relation such that H/γ^* is an abelian group. Also, in Theorem 3.3 of [7], Freni proved that in every hypergroup, the relation γ is transitive, that is $\gamma^* = \gamma$, and in this case, according to Corollary 1.2 of [7], the quotient H/γ^* is an abelian group. The γ^*-relation is, in some sense, a generalization of the β^*-relation.

The letter γ already has been used for the corresponding fundamental relation on hyperrings by Vougiouklis [12,13]. Thus, there is a confusion on the symbolism. similar to [6] we use the symbol Γ instead of γ for hyperrings. Vougiouklis in [12] defined the fundamental relation Γ^* on hyperring R as the smallest equivalence relation on R such that the quotient R/Γ^* is a fundamental ring. Let $(R, +, \circ)$ be a hyperring. Vougiouklis defined the relation Γ as follows:

> $a\Gamma b$ if and only if $\{a, b\} \subseteq u$, where u is a finite sum of finite products of elements of R (u may be a sum of only one element), in fact there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$ such that $u = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$.

He proved that Γ^* is the transitive closure of Γ. The both $\oplus$ and $\odot$ on R/Γ^* are defined as follows:

$$\begin{array}{ll}\Gamma^*(a) \oplus \Gamma^*(b) = \Gamma^*(c) & \text{for all } c \in \Gamma^*(a) + \Gamma^*(b), \\ \Gamma^*(a) \odot \Gamma^*(b) = \Gamma^*(d) & \text{for all } d \in \Gamma^*(a) \circ \Gamma^*(b).\end{array}$$

The commutativity of addition in rings can be related with the existence of the unit in multiplication. If e is the unit in a ring then for all elements a, b we have

$$\begin{array}{l}(a+b)(e+e) = (a+b)e + (a+b)e = a+b+a+b, \\ (a+b)(e+e) = a(e+e) + b(e+e) = a+a+b+b.\end{array}$$

So $a+b+a+b = a+a+b+b$ gives $b+a = a+b$. Therefore, when we say $(R, +, \circ)$ is a hyperring, $(+)$ is not commutative and there is not unit in the multiplication. So the commutativity, as well as the existence of the unit, it is not assumed in the fundamental ring. Of course, we know there exist many rings ($+$ is commutative) while don't have unit. With above notations we have:

Theorem 1.1. [12, 13] *Let $(R, +, \circ)$ be a hyperring and let Γ^* be the transitive closure of Γ.*

(1) Γ^* is a strongly regular relation both on $(R, +)$ and $(R, \circ)$.

(2) The quotient R/Γ^* is a ring.

(3) The relation Γ^* is the smallest equivalence relation such that the quotient R/Γ^* is a ring.

In [6], Davvaz and Vougiouklis introduced the relation α as a new strongly regular equivalence relation on a hyperring such that the set of quotients is an ordinary commutative ring. In this paper we need to follow concepts:

A non-empty subset A of a semi-hypergroup $(H, \circ)$ is called *invertible to the left* if for every $(x, y) \in H^2$, the implication $y \in A \circ x \Rightarrow x \in A \circ y$ is valid; that A is *invertible* if A is invertible to the right and to the left.

We say that a subhypergroup K of $(H, \circ)$ is *closed to the right* if for every $a \in H$ and for every $(x, y) \in K^2$, from $y \in x \circ a$ follows $a \in K$. We say that K is *closed* if it is closed to the right and the left.

2 Transitivity of Γ-relation

Definition 2.1. *Let M be a non-empty part of R. We say that M is a complete if for every $n \in \mathbb{N}$, $i = 1, 2, \ldots, n$, $\forall k_i \in \mathbb{N}$, $\forall (z_{i1}, z_{i2}, \ldots, z_{ik_i}) \in R^{n_i}$, we have*

$$\sum_{i=1}^{n} (\prod_{j=1}^{k_i} z_{ij}) \cap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} (\prod_{j=1}^{k_i} z_{ij}) \subseteq M.$$

Theorem 2.2. *Let ρ be a strongly regular equivalence relation on a hyperring $(R, +, \circ)$, for every $z \in R$, $\rho(z)$ be the class of z module ρ. Then, $\rho(z)$ is a complete part.*

Proof: Let $\sum_{i=1}^{n}(\prod_{j=1}^{k_i} z_{ij}) \bigcap \rho(z) \neq \emptyset$, then there exists $y \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} z_{ij}) \bigcap \rho(z)$. R/ρ is a hyperring and $\varphi : R \to R/\rho$ is the canonical projection, so

$$\varphi(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} z_{ij})) = \varphi(y) = \rho(z),$$

thus $\sum_{i=1}^{n}(\prod_{j=1}^{k_i} z_{ij}) \subseteq \rho(z)$. □

Definition 2.3. *Let M be a non-empty part of a hyperring R. The intersection of the parts of R which are complete and contain M is called the complete closure of M in R; it will be denoted by $\mathcal{C}(M)$.*

Before proving the next theorem, we consider the following notation:

$$[z]^n_{k_1,\ldots,k_n} = \{(z_{i1}, z_{i2}, \ldots, z_{ik_i}) \in R^{k_i} | i = 1, 2 \ldots, n\}.$$

Theorem 2.4. *Let M be a non-empty part of a hyperring R. We set:*

$K_1(M) = M$,

$K_{n+1}(M) = \{x \in R | \exists [z]^n_{k_1,\ldots,k_n} : z \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} z_{ij}),\ \sum_{i=1}^{n}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_n(M) \neq \emptyset\}$,

$K(M) = \bigcup_{n\geq 1} K_n(M)$.

Then, $K(M) = \mathcal{C}(M)$.

Proof: It is necessary to prove that:

1) $K(M)$ is a complete part of R;

2) If $M \subseteq N$ and N is complete, then $K(M) \subseteq N$.

1)$\sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K(A) \neq \emptyset$ implies that $n \in \mathbb{N}$ exists such that

$$\sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_n(A) \neq \emptyset,$$

from follows $\sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \subseteq K_{n+1} \subseteq K(A)$.

2) $K_1(M) = M \subseteq N$. Suppose that $K_n(M) \subseteq N$, we prove that this implies $K_{n+1}(M) \subseteq N$. In fact if $z \in K_{n+1}(M)$, there exists $[z]^p_{k_1,\ldots,k_n}$ such that $z \in \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij})$ and $\sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_n(M) \neq \emptyset$. Then

$$K_n(M) \subseteq N \Rightarrow \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap N \neq \emptyset,$$

from $\sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \subseteq N$, for this reason $z \in B$ and thus $K_{n+1}(M) \subseteq B$. □

Lemma 2.5. *1) For every $n \geq 2$ and $x \in R$, we have $K_n(K_2(x)) = K_{n+1}(x)$,*

2) $x \in K_n(y) \;\Leftrightarrow\; y \in K_n(x)$.

Proof: 1) The proof is by induction: If $n = 2$ we have

$$K_2(K_2(x)) = \{z | \exists [z]^n_{k_1,\ldots,k_n}, z \in \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}), \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_2(x) \neq \emptyset\} = K_3(x).$$

Now, suppose that $K_{n-1}(K_2(x)) = K_n(x)$; then:

$$\begin{aligned} K_n(K_2(x)) &= \{z | \exists [z]^p_{k_1,k_2,\ldots,k_p}, z \in \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}), \\ &\quad \textstyle\sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_{n-1}(K_2(x)) \neq \emptyset\} \\ &= \{z | \exists [z]^p_{k_1,k_2,\ldots,k_p}, z \in \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}), \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_n(x) \neq \emptyset\} \\ &= K_{n+1}(x). \end{aligned}$$

2) We also prove (2) by induction. It is clear that $x \in K_2(y) \;\Leftrightarrow\; y \in K_2(x)$. Suppose $x \in K_{n-1}(y) \Leftrightarrow y \in K_{n-1}(x)$. Let $x \in K_n(y)$, then there exists $[z]^p_{k_1,\ldots,k_n}$ such that $z \in \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}), \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_{n-1}(y) \neq \emptyset$, from this there exists $v \in \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_n(y)$. Since $x, v \in \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij})$ we have $v \in K_2(x)$. From $v \in K_{n-1}(y)$ we have $y \in K_{n-1}(v)$. Now, we obtain $y \in K_{n-1}(K_2(x)) = K_n(x)$.
□

Theorem 2.6. *We define the relation K as follows:*

$$xKy \Leftrightarrow x \in \mathcal{C}(y),$$

then K is an equivalence relation.

Proof: Since $x \in \mathcal{C}(x)$, the relation K is reflexive. By Lemma 2.5, K is symmetric. Now, let xKy and yKz. If P is a complete part and $z \in P$ then $\mathcal{C}(z) \subseteq P$, thus $y \in P$ and consequently $\mathcal{C}(y) \subseteq P$ and $x \in \mathcal{C}(y)$. Thus, $x \in \mathcal{C}(z)$, and so xKz. □

Theorem 2.7. *For every $(x, y) \in R^2$, we have xKy if and only if $x\Gamma^*\, y$.*

Proof: Suppose $x\Gamma y$ then $x \in K_2(y)$, therefore $x \in K(y)$. Thus $\Gamma \subseteq K$, since K is an equivalence relation we have $\Gamma^* \subseteq K$.

Conversely if xKy then there exists $n \in N$ such that $x \in K_{n+1}(y)$, so there exists $\sum\limits_{i=1}^{p}(\prod\limits_{j=1}^{k_i} z_{ij})$ which contains x and $\sum\limits_{i=1}^{p}(\prod\limits_{j=1}^{k_i} z_{ij}) \bigcap K_n(y) \neq \emptyset$. Thus there exists $x_1 \in \sum\limits_{i=1}^{p}(\prod\limits_{j=1}^{k_i} z_{ij}) \bigcap K_n(y)$. Then, $\{x_1, x_2\} \subseteq \sum_{i=1}^{p}(\prod_{j=1}^{k_i} z_{ij})$ and so $x_1\Gamma x$. Now, we have $x_1 \in K_n(y)$ and the similar way implies there exists $x_2 \in R$ such that $x_1\Gamma x_2$. So a consequence one attains $x_3, \ldots, x_n$ such that $x_n \in K_1(y)$ and $x\Gamma x_1\Gamma \ldots \Gamma x_n = y$. Therefore, $\Gamma^* \subseteq K$. □

Theorem 2.8. *If B is a non-empty part of R, then we have $\mathcal{C}(B) = \bigcup\limits_{b \in B} \mathcal{C}(b)$.*

Proof: It is clear for every $b \in B$, $\mathcal{C}(b) \subseteq \mathcal{C}(B)$ and so $\bigcup\limits_{b \in B} \mathcal{C}(b) \subseteq \mathcal{C}(B)$. To prove the converse we have by Theorem 2.4, $\mathcal{C}(B) = \bigcup\limits_{n \geq 1} K_n(B)$.

Now, we prove by induction $K_n(B) = \bigcup\limits_{b \in B} K_n(b)$. If n=1 then $K_1(B) = B = \bigcup\limits_{b \in B} \{b\} = \bigcup\limits_{b \in B} K_n(b)$. Now, suppose it is true for n, that is, $K_n(B) \subseteq \bigcup\limits_{b \in B} K_n(b)$ and we prove that $K_{n+1}(B) \subseteq \bigcup\limits_{b \in B} K_{n+1}(b)$. If $z \in K_{n+1}(B)$, then there exists $[z]^q_{k_1,\ldots,k_n}$ such that $z \in \sum\limits_{i=1}^{q}(\prod\limits_{j=1}^{k_i} z_{ij})$ and $\sum\limits_{i=1}^{q}(\prod\limits_{j=1}^{k_i} z_{ij}) \bigcap K_n(B) \neq \emptyset$. From this and hypothesis induction we have $\sum\limits_{i=1}^{q}(\prod\limits_{j=1}^{k_i} z_{ij}) \bigcap (\bigcup\limits_{b \in B} K_n(b)) \neq \emptyset$. Therefore,

$b' \in B$ exists such that $\sum_{i=1}^{q}(\prod_{j=1}^{k_i} z_{ij}) \bigcap K_n(b') \neq \emptyset$. Since $z \in \sum_{i=1}^{q}(\prod_{j=1}^{k_i} z_{ij})$ then $z \in K_{n+1}(b')$ and so $K_{n+1}(B) \subseteq \bigcup_{b\in B} K_{n+1}(b)$. Thus,

$$\mathcal{C}(B) = \bigcup_{n\geq 1} K_n(B) \subseteq \bigcup_{n\geq 1}\bigcup_{b\in B} K_n(b)) = \bigcup_{b\in B}\bigcup_{n\geq 1} K_n(b)) = \bigcup_{b\in B} \mathcal{C}(b).$$

□

Let $(R, +, \circ)$ and $(R', \dagger, \star)$ be two hyperrings, we say that $f : R \to R'$ is a homomorphism if for every $(x, y) \in R^2$ we have $f(x + y) = f(x) \dagger f(y)$ and $f(x \circ y) = f(x) \star f(y)$.

Definition 2.9. *Let R/Γ^* be a ring with unitary element and $\varphi_R : R \to R/\Gamma^*$ be the canonical projection, we set $\omega_R = \varphi_R^{-1}(1_{R/\Gamma^*})$, and ω_R is called the heart of R*

Theorem 2.10. *If $(R, +, \circ)$ is a hyperfield, i.e., $(R, +, \circ)$ is a hyperring and $(R, \circ)$ is a hypergroup, and B is a part of R, then we have $\omega_R \circ B = B \circ \omega_R = \varphi_R^{-1}(\varphi_R(B))$.*

Proof: Clearly $\varphi_R^{-1}(\varphi_R(B)) = \{x \in R | \exists b \in B : \varphi_R(b) = \varphi_R(x)\}$. Let $y \in \varphi_R^{-1}(\varphi_R(B))$, thus for some $b \in B$, $\varphi_R(y) = \varphi_R(b)$. Since $(R, \circ)$ is a hypergroup, $u \in R$ exists such that $y \in b \circ u$, so $\varphi_R(y) = \varphi_R(b) \odot \varphi_R(u)$. Since $(R/\Gamma^*, \odot)$ is a group and $\varphi_R(y) = \varphi_R(b)$, we obtain $\varphi_R(u) = 1_{R/\Gamma^*}$ and so $u \in \varphi^{-1}(1_{R/\Gamma^*}) = \omega_R$. Therefore, $\varphi_R^{-1}(\varphi_R(B)) \subseteq B \circ \omega_R$.

Conversely if $z \in B \circ \omega_R$, then $\varphi_R(z) = \varphi_R(B)$ and so $z \in \varphi_R^{-1}(\varphi_R(B))$. It is proved that $B \circ \omega_R = \varphi_R^{-1}(\varphi_R(B))$. By a similar way, we obtain $\omega_R \circ B = \varphi_R^{-1}(\varphi_R(B))$. □

Theorem 2.11. *If $(R, +, \circ)$ is a hyperfield and B is a part of R, then we have $\omega_R \circ B = B \circ \omega_R = \mathcal{C}(B)$.*

Proof: If $\varphi_R(b) = \varphi_R(x)$ by Theorem 2.6 and 2.7, $x \in \mathcal{C}(b)$. Therefore

$$\varphi_R^{-1}(\varphi_R(B)) = \{x \in R | \exists b \in B : x \in \mathcal{C}(b)\} = \bigcup_{b\in B} \mathcal{C}(b) = \mathcal{C}(B)$$

and by Theorem 2.11, the proof is completed. □

Corollary 2.12. *If A is a non-empty part of hyperfield $(R, +, \circ)$, then A is a complete part if and only if $A = A \circ \omega_R$.*

Proof: By Theorem 2.11, $\mathcal{C}(A) = A \circ \omega_R$. Now, $A = A \circ \omega_R$ if and only if $A = \mathcal{C}(A)$ and the proof is completed . □

Theorem 2.13. *If A is a complete part and B is a non-empty part of a hyperfield R, then $A \circ B$ and $B \circ A$ are complete.*

Proof: By Theorem 2.11, we have $\mathcal{C}(A \circ B) = A \circ B \circ \omega_R = A \circ \omega_R \circ B = A \circ B$. Thus, by Corollary 2.12, $A \circ B$ is a complete part. □

Theorem 2.14. *If $(R, +, \circ)$ is a hyperfield and A is a subhypergroup of $(R, \circ)$, then:*

1) If A is a complete part, then A is invertible (by hyperoperation $\circ$).

2) If A is invertible to the right, then it is closed to the left.

Proof: 1) Let $y \in A \circ x$, then $a \in A$ exists such that $y \in a \circ x$, from which $\varphi_R(y) = \varphi_R(a) \odot \varphi_R(x)$, from this follows $\varphi_R(x) = \varphi_R(a)^{-1} \odot \varphi_R(y)$, so since $\varphi_R(A)$ is subgroup of $(R/\Gamma^*, \odot)$ we have $\varphi_R(x) \in \varphi_R(A) \odot \varphi_R(y) = \varphi_R(A \circ y)$. By Theorem 2.13, $A \circ y$ is complete part. Therefore, $x \in \varphi_R^{-1}(\varphi_R(A \circ y)) = A \circ y$, and (2) is easily verified. □

Theorem 2.15. *The heart of hyperfield R is the intersection of all the subhypergroup of $(R, \circ)$ which are complete part.*

Proof: We set SC the set of all subhypergroups of $(R, \circ)$ which are complete parts. Since $\omega_R = \varphi^{-1}(1_{R/\Gamma^*})$, then ω_R is subhypergroup of $(R, \circ)$, and so $\omega_R \circ \omega_R = \omega_R$, thus by Theorem 2.12, ω_R is a complete part. It is sufficient to prove that if $A \in SC$, then $\omega_R \subseteq A$. Since $A \in SC$ by Theorem 2.14, A is invertible. On the other hand $A = A \circ \omega_R$. Hence, for every $x \in \omega_R$ there exists $(a, b) \in A^2$ such that $b \in a \circ x$, therefore $b \in A \circ x$. Since A is invertible thus $x \in A \circ b$. Therefore, $x \in A$ and this implies $\omega_R \subseteq A$. □

Before proving the next theorem, we introduce the following notations. For every element z of a hyperring R, we put:

$$P(z) = \{A \in P^*(R) | z \in A, \exists [x]^n_{k_1,\ldots,k_n}, A = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij})\}$$

and

$$M(z) = \bigcup_{A \in P(z)} A$$

Theorem 2.16. *For every $z \in R$, $M = M(z)$ is a complete part.*

Proof: Suppose that $\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \bigcap M \neq \emptyset$, then there exists

$$a \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \bigcap M$$

and so there exists $A \in P(z)$ such that $a \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \bigcap A$. Since $(R, \circ)$ is reproducible, there exist $w_1, \ldots, w_n \in R$ and $b \in R$ such that $x_{ik_i} \in w_i \circ z$ and $z \in a \circ b$. Now, $\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i-1} x_{ij} \circ w_i \circ z) \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i-1} x_{ij} \circ w_i \circ a \circ b) \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i-1} x_{ij} \circ w_i \circ A \circ b)$. Also we have $z \in a \circ b \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i-1} x_{ij} \circ x_{ik_i}) \circ b \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i-1} x_{ij} \circ w_i \circ z) \circ b \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i-1} x_{ij} \circ w_i \circ A \circ b)$. But $\sum_{i=1}^{n}(\prod_{j=1}^{k_i-1} x_{ij} \circ w_i \circ A \circ b) \in P(z)$ and therefore $\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \subseteq M(z)$. □

Corollary 2.17. *Let R be a hyperfield. Then, for every $z \in \omega_R$ we have $M(z) = \omega_R$.*

Proof: If $z \in \omega_R$, then for every $x \in M(z)$, by Theorem 2.16, we have $\omega_R = \mathcal{C}(z) \subseteq M(z)$. Conversely, it is clear that $M(z) \subseteq \omega_R$. Thus, $M(z) = \omega_R$. □

Theorem 2.18. *If R is a hyperfield then $\Gamma^* = \Gamma$*

Proof: If $x\ \Gamma^*\ y$, then there exists $(v, w) \in \omega_R^2$ such that $y \in x \circ v$ and $x \in x \circ w$. By Theorem 2.16, $M(w) = \omega_R$. Therefore, there exists $A \in P(w)$ such that $v \in A$. If $A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ then $\{x, y\} \subseteq x \circ \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and so $x \Gamma y$. □

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Department of Mathematics, Yazd University,
Yazd, Iran
E-mail: saeed_mirvakili@yahoo.com
E-mail: anvariyeh@yazduni.ac.i
E-mail: davvaz@yazduni.ac.ir