Bull. Math. Soc. Sc. Math. Roumanie
Tome 48(96) No. 4, 2005, 361–367

On a class of differential inclusions governed by a sweeping process

by
AURELIAN CERNEA, VASILE LUPULESCU

Abstract

We prove the existence of local solutions for a class of evolution inclusions defined by a sweeping process and by a set-valued map with nonconvex values.

Key Words: Sweeping process, normal cone, ϕ-convex function of order two.
2000 Mathematics Subject Classification: Primary: 34A60.

1 Introduction

The existence of local solutions to evolution equations governed by differential inclusions and by sweeping processes has been the subject of many papers in the last two decades. Convex sweeping process were introduced by Moreau ([8]). We refer to [4] and [7] for a complete bibliography on this topic.

The aim of the present note is to establish an existence result for nonconvex perturbations of the sweeping process associated with a closed convex locally compact set C of a Hilbert space

$$x'(t) \in -N_C(x(t)) + F(x(t)) + g(t) \quad a.e.\ ([0,T]), \quad x(0) = x_0, \tag{1.1}$$

where $N_C(x(t))$ denotes the Clarke normal cone to C at $x(t)$, $F(.)$ is a set-valued map, upper semicontinuous on H with nonempty compact values satisfying $F(x) \subset \partial_F V(x)$, $\forall x \in H$ with $V(.)$ a ϕ-convex function of order two, $\partial_F V(.)$ is the Fréchet subdifferential of $V(.)$ and $g(.)$ is a bounded measurable function on H.

Our result is an improvement of a previous result of Syam ([9]). In [9] the set-valued map $F(.)$ is assumed to satisfy $F(x) \subset \partial V(x)$, $\forall x \in H$ with $V(.)$ a convex function and $\partial V(.)$ denotes the subdifferential (in the sense of Convex Analysis). Since the class of proper convex functions is strictly contained into the class of ϕ- convex functions of order two, our result generalizes the one in [9]. On the other hand, our result may be interpreted as an extension of a result of Cardinali, Colombo, Papalini and Tosques ([3]).

The proof of our main result follows the general ideas in [3] and [9].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

Let H be a real Hilbert space with the norm $||.||$ and the scalar product $\langle .,. \rangle$. We denote by B the closed unit ball in H. For $x \in H$ and for a closed subset $A \subset X$ we denote by $\mathrm{d}(x, A)$ the distance from x to A given by $\mathrm{d}(x, A) := \inf\{||y-x||; y \in A\}$. By $co(A)$ we denote the convex hull of A and by $\overline{co}(A)$ we denote the closed convex hull of A

If $K \subset H$ is a closed set and $x \in K$, *Clarke's tangent cone* to K at x is defined by

$$C_K(x) = \{v \in H; \quad \lim_{s \to 0+, x' \to_K x} \frac{d(x' + sv, K)}{s} = 0\},$$

where $\to_K$ denotes the convergence in K. The negative polar of Clarke's tangent cone $N_K(x) := C_K(x)^-$ is also called the *normal cone* to the set K at $x \in K$.

Let $\Omega \subset H$ be an open set and let $V : \Omega \to \mathbf{R} \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in \Omega;\ V(x) < +\infty\}$.

Definition 2.1 We call *Fréchet subdifferential* of V the multifunction $\partial_F V : \Omega \to \mathcal{P}(H)$, defined by

$$\partial_F V(x) = \{\alpha \in H, \liminf_{y \to x} \frac{V(y) - V(x) - < \alpha, y - x >}{||y - x||} \geq 0\} \quad \text{if}\, V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$.

Define $D(\partial_F V) = \{x \in H; \partial_F V(x) \neq \emptyset\}$.

According to [6] the values of $\partial_F V(.)$ are closed and convex.

Definition 2.2 Let $V : \Omega \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous function. We say that V is a *ϕ-convex of order 2* if there exists a continuous map $\phi_V : (D(V))^2 \times \mathbf{R}^2 \to \mathbf{R}_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$V(y) \geq V(x) + < \alpha, x - y > -\phi_V(x, y, V(x), V(y))(1 + ||\alpha||^2)||x - y||^2. \quad (2.1)$$

In [3], [6] there are several examples and properties of such maps. For example, according to [3], if $K \subset \mathbf{R}^2$ is a closed and bounded domain, whose boundary is a C^2 regular Jordan curve, the indicator function of K

$$V(x) = I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K \\ +\infty, & \text{otherwise} \end{cases}$$

is ϕ- convex of order 2.

The next result, due to Syam, will be used in the proof of our main result.

Proposition 2.3 ([9]) *Consider $C \subset H$ a nonempty convex set and $x_0 \in C$. Then for any $g(.) \in L^1([0,T],H)$ there exists an unique absolutely continuous function $x_g(.);[0,T] \to H$ solution to*

$$x'(t) \in -N_C(x(t)) + g(t) \quad a.e. \ ([0,T]), \quad x(0) = x_0.$$

Moreover,

$$< g(t) - x_g'(t), x_g'(t) >= 0 \quad a.e. \ ([0,T]), \tag{2.2}$$

$$||x_g'(t)|| \leq 2||g(t)|| \quad a.e. \ ([0,T]). \tag{2.3}$$

3 The main result

We are now able to prove the main result of this paper.

Theorem 3.1 *Let $F(.) : H \to \mathcal{P}(H)$ be an upper semicontinuous set-valued map with nonempty compact values such that there exists a locally Lipschitz ϕ-convex of order two function $V(.) : H \to \mathbf{R}$ with $F(x) \subset \partial_F V(x)$, $\forall x \in H$. Consider $g(.) : [0,\infty) \to H$ a bounded measurable function and $C \subset H$ a closed convex locally compact set.*

Then for any $x_0 \in C$ there exist $T > 0$ and an absolutely continuous function $x(.) : [0,T] \to H$ solution to problem (1.1).

Proof: Let $r, L > 0$ be such that $V(.)$ is L-Lipschitz on $x_0 + rB$. Using the properties of the Fréchet subdifferential we have that $\partial_F V(x) \subset LB$ $\forall x \in rB$. Without loss of generality we assume that $||g(t)|| \leq 1$ $\forall t \in [0,\infty)$.

We take $T > 0$ such that $T < \frac{r}{2(L+1)}$, we denote $I = [0,T]$ and we define $t_n^i = \frac{i}{2^n}T$, $i = 0,1,...,2^n$. Consider ϕ_v the function appearing in Definition 2.2 and

$$M := \{\phi_W(x_1,x_2,y_1,y_2), x_i \in x_0 + rB, y_i \in [V(x_0) - 1, V(x_0) + 1], i = 1,2\} <$$

$+\infty$.

Take $y_0 \in F(x_0)$ and define $f_1^n(.) : [0,t_1^n] \to H$ by $f_1^n(t) \equiv y_0$. Obviously, $f_1^n(.) \in L^2([0,t_1^n],H)$.

Applying Proposition 2.3 we obtain the existence of an unique absolutely continuous function $x_1^n(.) : [0,t_1^n] \to H$ solution to

$$x'(t) \in -N_C(x(t)) + f_1^n(t) + g(t) \quad a.e. \ ([0,t_1^n]), \quad x(0) = x_0.$$

Since, for any $t \in [0,t_1^n]$, $||f_1^n(t)|| \leq L$ we deduce

$$||(x_1^n)'(t)|| \leq 2||f_1^n(t) + g(t)|| < 2(L+1) \quad a.e. \ ([0,t_1^n]),$$

thus $||x_1^n(t)-x_0|| \leq 2^{-n}2(L+1)T < r\ \forall t \in [0,t_1^n]$, i.e., $x_1^n(t) \in x_0+rB\ \forall t \in [0,t_1^n]$.

Repeating the same construction for any $k \in \{2,3,...,2^n\}$ we take $y_{k-1}^n \in F(x_{k-1}^n(t_{k-1}^n))$ with $x_{k-1}^n(t_{k-1}^n) \in x_0+rB$ and define $f_k^n(.) : (t_{k-1}^n, t_k^n] \to H$ by $f_k^n(t) \equiv y_{k-1}^n$. By Proposition 2.3 we consider the unique solution $x_k^n(.) : [t_{k-1}^n, t_k^n] \to H$ to

$$x'(t) \in -N_C(x(t)) + f_k^n(t) + g(t) \quad a.e.\ ([t_{k-1}^n, t_k^n]), \quad x(t_{k-1}^n) = x_{k-1}^n(t_{k-1}^n).$$

One has $||(x_k^n)'(t)|| \leq 2||f_k^n(t)+g(t)|| < 2(L+1)$ and therefore, for any $t \in [t_{k-1}^n, t_k^n]$

$$||x_k^n(t) - x_k^n(t_{k-1}^n)|| \leq 2(L+1)(t-t_{k-1}^n) \leq 2^{-n}2(L+1)T < r$$

Define

$$\theta_n(t) = t_{k-1}^n \quad \forall t \in [t_{k-1}^n, t_k^n), \quad k=1,2,...,2^n, \quad \theta_n(T) = T,$$

$$x_n(t) = \sum_{k=1}^{2^n} x_k^n(t)\chi_{[t_{k-1}^n, t_k^n]}(t), \quad f_n(t) = \sum_{k=1}^{2^n} f_k^n(t)\chi_{(t_{k-1}^n, t_k^n]}(t),$$

where χ_A is the characteristic function of the set A.

Then, for any $t \in [t_{k-1}^n, t_k^n]$,

$$||x_n(t)-x_0|| \leq ||x_n(t)-x_n(t_{k-1}^n)|| + ||x_n(t_{k-1}^n) - x_n(t_{k-2}^n)|| + ... + ||x_n(t_1^n) - x_0||$$
$$\leq 2k(L+1)2^{-n}T < 2^n.2(L+1)2^{-n}T = 2(L+1)T < r.$$

i.e., $x_n(t) \in x_0 + rB\ \forall t \in I$.

Therefore, we have

$$x_n(t) \in C \cap (x_0+rB) \quad \forall t \in I, \tag{3.1}$$

$$x_n'(t) \in -N_C(x_n(t)) + f_n(t) + g(t) \quad a.e.\ (I), \tag{3.2}$$

$$f_n(t) \in F(x_n(\theta_n(t))) \subset \partial_F V(x_n(\theta_n(t))) \subset LB \quad a.e.\ (I) \tag{3.3}$$

and, via Proposition 2.3, one has

$$||x_n'(t)|| \leq 2(||g(t)|| + ||f_n(t)||) \leq 2(L+1) \quad a.e.\ (I), \tag{3.4}$$

$$< x_n'(t), x_n'(t) > = < x_n'(t), g(t)+f_n(t) > \quad a.e.\ (I). \tag{3.5}$$

We prove next

$$\int_0^T < f_n(t), x_n'(t) > dt \leq V(x_n(T)) - V(x_0) + M(1+L^2)\frac{T}{2^n}\int_0^T ||x_n'(t)||^2 dt. \tag{3.6}$$

Indeed, using the properties of the function $V(.)$, we have

$$V(x_n(t_k^n)) \geq V(x_n(t_{k-1}^n)) + < y_{k-1}^n, x_k^n - x_{k-1}^n > -$$

$$\phi_v(x_n(t_{k-1}^n), x_n(t_k^n), V(x_n(t_{k-1}^n)), V(x_n(t_k^n)))(1 + ||y_{k-1}^n||^2)||x_k^n - x_{k-1}^n||^2.$$

So,

$$< y_{k-1}^n, \int_{t_{k-1}^n}^{t_k^n} x_n'(t)dt > \leq V(x_n(t_k^n)) - V(x_n(t_{k-1}^n)) +$$

$$\phi_v(x_n(t_{k-1}^n), x_n(t_k^n), V(x_n(t_{k-1}^n)), V(x_n(t_k^n)))(1 + ||y_{k-1}^n||^2)|| \int_{t_{k-1}^n}^{t_k^n} x_n'(t)dt||^2.$$

We deduce

$$\int_{t_{k-1}^n}^{t_k^n} < f_n(t), x_n'(t) > dt \leq$$

$$\leq V(x_n(t_k^n)) - V(x_n(t_{k-1}^n)) + M(1 + L^2)\frac{T}{2^n} \int_{t_{k-1}^n}^{t_k^n} ||x_n'(t)||^2 dt,$$

hence (3.6) holds true.

From (3.4) and Theorem III. 27 in [2] we deduce the existence of a subsequence (again denoted) $x_n'(.)$ which converges weakly $L^1(I, H)$ to a function $y(.) \in L^1(I, H)$. In particular, $\lim_{n\to\infty}(x_0 + \int_0^t x_n'(s)ds) = x_0 + \int_0^t y(s)ds \ \forall t \in I$.

On the other hand, from (3.1) and the fact that the set $C \cap (x_0 + rB)$ is convex we find that $x_n(.)$ converges uniformly to $x(.)$, where $x(t) := x_0 + \int_0^t y(s)ds \ \forall t \in I$.

Using the continuity of $V(.)$, (3.4), (3.6) one has

$$\limsup_{n\to\infty} \int_0^T < f_n(t), x_n'(t) > dt \leq V(x(T)) - V(x_0). \tag{3.7}$$

At the same time, from (3.3) and Theorem III. 27 in [2] there exists a subsequence (again denoted) $f_n(.)$ which converges weakly $L^1(I, H)$ to a function $f(.) \in L^1(I, H)$ and thus, since $x_n(\theta_n(.))$ converges uniformly to $x(.)$ and $\overline{co}F(.)$ is upper semicontinuous with compact convex values, we apply Theorem 1.4.1 in [1] and we find that

$$f(t) \in \overline{co}F(x(t)) \subset \partial_F V(x(t)) \quad a.e. \ (I). \tag{3.8}$$

Next we apply Theorem 2.2 in [3] and we deduce from (3.8) that

$$\frac{d}{dt}(V \circ x)(t) =< x'(t), f(t) > \quad a.e. \ (I) \tag{3.9}$$

which implies

$$V(x(T)) - V(x_0) = \int_0^T < x'(t), f(t) > dt. \tag{3.10}$$

By a standard argument (e.g. [9]) from (3.2), from the weakly convergence of $x_n'(.)$ to $x'(.)$ in $L^2(I, H)$, from the weakly convergence of $f_n(.)$ to $f(.)$ in $L^2(I, H)$ and the uniform convergence of $x_n(.)$ to $x(.)$ we have

$$x'(t) \in -N_C(x(t)) + f(t) + g(t) \quad a.e. \ (I). \tag{3.11}$$

Using Proposition 2.3 we have

$$< x'(t), x'(t) >=< x'(t), g(t) + f(t) > \quad a.e.\ (I). \tag{3.12}$$

On the other hand, from (3.5), (3.7) and (3.10) we have

$$\limsup_{n\to\infty} \int_0^T < x_n'(t), x_n'(t) > dt = \lim_{n\to\infty} \int_0^T < x_n'(t), g(t) > dt + \limsup_{n\to\infty}$$

$$\int_0^T < x_n'(t), f_n(t) > dt \leq \int_0^T < x'(t), g(t) > dt + \int_0^T < x'(t), f(t) > dt. \tag{3.13}$$

From (3.12) and (3.13) we infer that

$$\limsup_{n\to\infty} \int_0^T < x_n'(t), x_n'(t) > dt \leq \int_0^T < x'(t), x'(t) > dt.$$

By the weak lower semicontinuity of the norm (e.g. Prop. III. 30 in [2]) we deduce that $x_n'(.)$ converges to $x'(.)$ in the strong topology of $L^2(I,H)$. Therefore there exists a subsequence (again denoted) $x_n'(.)$ which converges pointwise a.e. in I to $x'(.)$.

It remains to prove that

$$x'(t) \in -N_C(x(t)) + F(x(t)) + g(t) \quad a.e.\ (I). \tag{3.14}$$

Define $X(t) := cl\{x_n(\theta_n(t)); n \in \mathbf{N}\}$, $t \in I$. Obviously, $X(t) \subset H$ is compact; since $F(.)$ is upper semicontinuous with compact values, we infer that $F(X(t)) \subset H$ is compact.

Define $Y(t) := cl\{x_n'(t) - f_n(t) - g(t); n \in \mathbf{N}\} \cup \{0\}$ for almost all $t \in I$ and $G(t,x) := -N_C(x) \cap Y(t)$, $x \in H$. Then $G(t,.)$ is upper semicontinuous on $C \cap (x_0 + rB)$ with compact values because the set-valued map $-N_C(.)$ has closed graph and $Y(t) \subset H$ is compact (e.g. Theorem 1.1.1 in [1]).

Since for almost all $t \in I$, $-x_n'(t) + f_n(t) + g(t) \in -G(t, x(t))$ one has

$$d(x_n'(t), G(t,x(t)) + F(x(t)) + g(t)) \leq d^*(G(t, x_n(t)) + F(x_n(\theta_n(t))) + g(t),$$

$$G(t,x(t)) + F(x(t)) + g(t)) \leq d^*(G(t,x_n(t)), G(t,x(t)) + d^*(F(x_n(\theta_n(t))),$$

$F(x(t)))$, where $d^*(A,B) = sup\{d(a,B), a \in A\}$.

From the upper semicontinuity of $G(t,.)$ anf $F(.)$ it follows

$$\lim_{n\to\infty} d(x_n'(t), G(t,x(t)) + F(x(t)) + g(t)) = 0 \quad a.e.\ (I)$$

and by the fact that $x_n'(t)$ converges to $x'(t)$ for almost all $t \in I$ we obtain

$$x'(t) \in G(t,x(t)) + F(x(t)) + g(t) \quad a.e.\ (I),$$

i.e., (3.14) holds true and the proof is complete. □

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Received: 10 05. 2005

Faculty of Mathematics and Informatics,
University of Bucharest,
Academiei 14,
010014 Bucharest, Romania.
E-mail: `acernea@math.math.unibuc.ro`

"Constantin Brâncuşi" University,
Republicii 1, 210152
Târgu Jiu, Romania.
E-mail: `vasile@utgjiu.ro`